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Multi-value cellular automaton models and metastable states in a congested phase

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Abstract. In this paper, a family of multi-value cellular automaton (CA) associated with traffic flow is presented. The family is obtained by extending the rule-184 CA, which is an ultradiscrete analogue to the Burgers equation. CA models in the family show both metastable states and stop-and-go waves, which are often observed in real traffic flow. Metastable states in the models exist not only on a prominent part of a free phase but also in a congested phase.

1. Introduction

Traffic phenomena have attracted much attention from physicists in recent years. They show a complex phase transition from the free to congested state, and many theoretical models have been proposed so far [1–7]. Among them we will focus on deterministic cellular automaton (CA) models. CA models are simple, flexible, and suitable for computer simulations of discrete phenomena.

The rule-184 CA [8] has been widely used as a prototype of a deterministic model of traffic flow. In the model, the lane is single and cars can move by one site at most every time step. Several variations of it have been proposed recently: first, Fukui and Ishibashi (FI) proposed a high-speed extension [6] of the rule-184 CA. In this model, cars can move more than one site per unit time if there are successive vacant spaces ahead. Second, Fukś and Boccara proposed a 'monitored traffic model' [9], which is a kind of quick-start (QS) model. In this model, drivers can prospect vacant spaces due to car motion in the next site and can move more quickly compared with the rule-184 CA. Third, Takayasu and Takayasu proposed a slow-start (SlS) model [10], in which cars cannot move just after they stop and must wait for a unit time. This represents an asymmetry of stopping and starting behaviour. We call all these variants a family of rule-184 CA in this paper.

A desirable condition for a CA traffic model is that it can show a phase transition observed in real data. Let us look at an example of real data from the Tomei expressway taken by the Japan Highway Public Corporation [12]. Figure 1 shows flow–density diagrams, often called 'fundamental diagrams', of both lanes in January 1996 on the up line at a point 170.64 km from Tokyo. Figures 1(a) and (b) present diagrams of the driving and acceleration lane, respectively. We see a phase transition from the free to congested state at around a density of 25 vehicle/km in both lanes, and there is a clearer discontinuity in the acceleration lane.

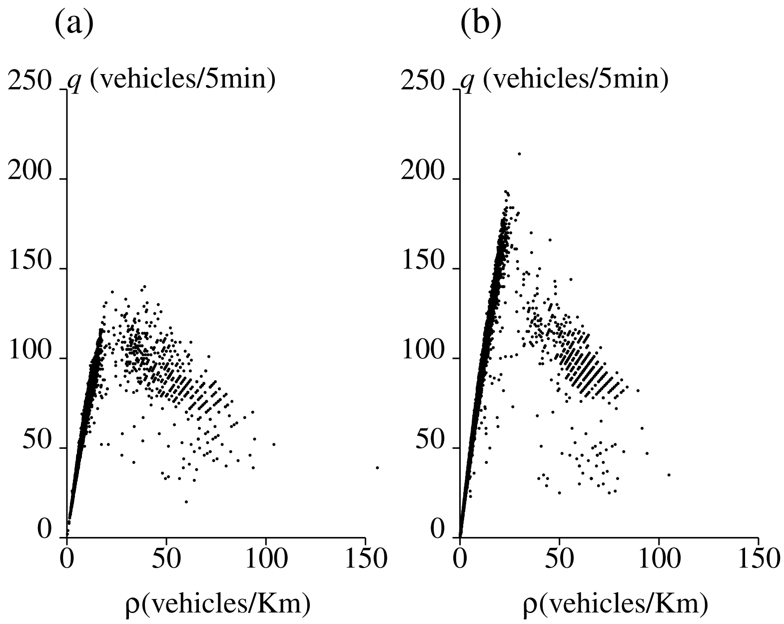

Figure 1. Observed data of flow (vehicle/5 min) versus density (vehicle/km) on the Tomei expressway. This diagram was taken by Japan Highway Public Corporation. (*a*) Driving lane, (*b*) acceleration lane.

The multiple states of flow at the same density around the critical density are also observed, particularly in the acceleration lane, and a skeleton curve of plot points shows a shape of 'inverse λ'. These phenomena are often observed in real data [12]. It is noted that a congested phase is represented by a two-dimensional region in figure 1, while a free phase is a thin line.

There is another observed fact that, over a certain critical density, a perturbation to uniform traffic causes formation of a jam, and a 'stop-and-go' wave propagates backward [13]. Cars accelerate and decelerate alternately in this wave. We adopt these two experimental facts, multiple states and stop-and-go waves, as criteria to judge whether a CA is suitable for a traffic model or not. Among the rule-184 CA family, only the SlS model shows multiple states and also shows a stop-and-go wave.

Recently, a multi-value generalization of the rule-184 CA has been proposed by using an ultradiscrete method [14]. Its evolution equation is

$$U_j^{t+1} = U_j^t + \min(U_{j-1}^t, L - U_j^t) - \min(U_j^t, L - U_{j+1}^t) \tag{1}$$

where U_j^t represents the number of cars at site j and time t, and L is an integer constant. Each site is assumed to hold L cars at most. We can prove that, if $L > 0$ and $0 \leqslant U_j^t \leqslant L$ for any j, then $0 \leqslant U_j^{t+1} \leqslant L$ holds for any j. Thus (1) is considered to be a $(L+1)$-value CA. Since (1) is obtained from an ultradiscretization of the Burgers equation, we call it the Burgers cellular automaton (BCA) [15]. Note that BCA is equivalent to the rule-184 CA in a special case of $L = 1$. The positive integer L can be interpreted physically in the following three ways: first, the road is an L-lane freeway in a coarse sense, and the effect of the lane-changing rule is not expressed explicitly. In the appendix, we show the BCA with $L = 2$ can be interpreted as a two-lane model of the rule-184 CA with an explicit lane-changing rule. Second, U_j^t/L represents a probability distribution of cars in a single-lane freeway. In this case, U_j^t itself no longer represents a real number of cars at site j. Third, also in a single-lane freeway, the length of each site is assumed to be long enough to contain L cars. By introducing the free parameter L into the rule-184 CA family, we can generalize them to multi-value CA and obtain rich algebraic properties and wide application to various transport phenomena.

Though BCA does not show multiple states at a density in the fundamental diagram, we have shown that its high-speed extension shows multiple states around a critical density [16]. Metastable states of flow in the model exist on a prominent part of a line in a free phase over a critical density. The model is also considered to be a multi-value generalization of the FI model. This model is mentioned in section 2.3 and compared with other models.

Other models in the rule-184 CA family, the SlS and QS models, are two-value CA, like the FI model. We generalize them to multi-value ones and investigate their properties as a traffic model in detail.

We use a max-plus representation to express the time evolution rule of those generalized models. This representation comes from the ultradiscrete method which has a close relation with the max-plus algebra [11], as shown in our previous papers. Using the max-plus representation, we can express the evolution rule of the above CAs in a conservation form like (1), which automatically derives the conservation of the total number of cars.

2. Generalization to multi-value CA

In this section, we will present four CA models which are multi-value generalizations of the rule-184 CA family. Before explaining the models, we briefly review a rule of car movement of BCA. Note that we assume L is a lane number of the road in the description of all models. In BCA, cars at site j move to vacant spaces at their next site $j+1$. Therefore, car flow from j to $j+1$ at time t becomes $\min(U_j^t, L-U_{j+1}^t)$. Thus adding $\min(U_{j-1}^t, L-U_j^t)$ (flow from $j-1$ to j) and $-\min(U_j^t, L-U_{j+1}^t)$ (flow from j to $j+1$) to U_j^t (the present car number), we obtain U_j^{t+1} and derive (1).

2.1. Multi-value QS model

First, we generalize the QS model to a multi-value one. In the multi-value model, cars at site j move to vacant spaces at site $j+1$ per unit time. This movement rule is similar to that of BCA. However, a difference is that drivers at j estimate vacant spaces at $j+1$ by predicting how many cars move from $j+1$ to $j+2$. Thus, they estimate vacant spaces at $j+1$ at time t to be $L-U_{j+1}^t$(present spaces)$+\min(U_{j+1}^t, L-U_{j+2}^t)$(predicted spaces due to BCA movement). Therefore, considering the number of cars coming into and escaping from site j, the evolution equation on U_j^t is given by

$$\begin{aligned} U_j^{t+1} = U_j^t &+ \min(U_{j-1}^t, L-U_j^t+\min(U_j^t, L-U_{j+1}^t)) \\ &- \min(U_j^t, L-U_{j+1}^t+\min(U_{j+1}^t, L-U_{j+2}^t)) \\ = U_j^t &+ \min(U_{j-1}^t, 2L-U_j^t-U_{j+1}^t) - \min(U_j^t, 2L-U_{j+1}^t-U_{j+2}^t). \end{aligned} \tag{2}$$

The final form of (2) means that the number of movable cars (U_j^t) is limited by the vacant spaces ($2L-U_{j+1}^t-U_{j+2}^t$) in their next two sites. In the case of $L=1$, this model is identical to the QS model proposed by Fukś and Boccara, and it is the rule-3212885888 CA of a neighbourhood size '5', after Wolfram's terminology [8].

2.2. Multi-value SlS model

In the original SlS model, the lane is single and cars cannot move just after they stop and must wait for a unit time. Movable cars move to vacant spaces in their next site as in BCA.

In the multi-value case, we should distinguish between standing cars and moving cars in each site because each site can hold many cars. The number of cars at site j blocked by cars at $j+1$ at time $t-1$ is represented by $U_j^{t-1} - \min(U_j^{t-1}, L - U_{j+1}^{t-1})$. These cars cannot move at t. Then the maximum number of cars movable to site $j+1$ at t is given by $U_j^t - \{U_j^{t-1} - \min(U_j^{t-1}, L - U_{j+1}^{t-1})\}$. Therefore, the multi-value SlS CA is given by

$$U_j^{t+1} = U_j^t + \min[U_{j-1}^t - \{U_{j-1}^{t-1} - \min(U_{j-1}^{t-1}, L - U_j^{t-1})\}, L - U_j^t] \\ - \min[U_j^t - \{U_j^{t-1} - \min(U_j^{t-1}, L - U_{j+1}^{t-1})\}, L - U_{j+1}^t]. \quad (3)$$

We note that this model includes the original SlS model for the case $L = 1$. Since U^{t+1} is determined by U^t and U^{t-1}, the evolution equation is second order in time and an effect of the inertia of cars is included in this model.

2.3. EBCA2 model

In our previous paper [16], we propose an extended Burgers cellular automaton (EBCA) model in which cars can move two sites forward at unit time if the successive two sites are not fully occupied. In this paper, we call the model 'EBCA2' and another model with a similar extension 'EBCA1' is described in the next section. A main difference between the two models is in the priority of movement of fast and slow cars to vacant spaces. Fast cars with speed 2 move prior to slow ones with speed 1 in EBCA2, whereas in EBCA1 they move *after* the slow ones.

The evolution equation of EBCA2 is given by

$$U_j^{t+1} = U_j^t + \min(b_{j-1}^t + a_{j-2}^t, L - U_j^t + a_{j-1}^t) - \min(b_j^t + a_{j-1}^t, L - U_{j+1}^t + a_j^t) \quad (4)$$

where $a_j^t \equiv \min(U_j^t, L - U_{j+1}^t, L - U_{j+2}^t)$ which is the number of cars moving by two sites and $b_j^t \equiv \min(U_j^t, L - U_{j+1}^t)$. In [16], we only study the cases of $L = 1$ and 2. In this paper, we show some properties of flow for L in the next section. Note that (4) in the case of $L = 1$ is the FI model, which is equivalent to the rule-3436170432 CA.

2.4. EBCA1 model

There is another possibility when we extend BCA to a high-speed model and we call this the extended new model EBCA1. In contrast to EBCA2, slow cars with speed 1 move prior to fast ones with speed 2. Then car movement from t to $t+1$ consists of the following two successive procedures:

(a) cars move to their next site if the site is not fully occupied;
(b) only cars moved in procedure (a) can move another one site if their next site is not fully occupied after procedure (a).

The number of moving cars at site j and time t in procedure (a) is given by $b_j^t \equiv \min(U_j^t, L - U_{j+1}^t)$. In procedure (b), the number of moving cars at site $j+1$ becomes $\min(b_j^t, L - U_{j+2}^t - b_{j+1}^t + b_{j+2}^t)$, where the second term in min represents vacant spaces at site $j+2$ after the first procedure (a). Therefore, considering the total number of cars entering into and escaping from site j, the evolution equation of EBCA1 is given by

$$U_j^{t+1} = U_j^t + b_{j-1}^t - b_j^t + \min(b_{j-2}^t, L - U_j^t - b_{j-1}^t + b_j^t) - \min(b_{j-1}^t, L - U_{j+1}^t - b_j^t + b_{j+1}^t) \\ = U_j^t + \min(b_{j-1}^t + b_{j-2}^t, L - U_j^t + b_j^t) - \min(b_j^t + b_{j-1}^t, L - U_{j+1}^t + b_{j+1}^t). \quad (5)$$

Note that (5) with $L = 1$ differs from the FI model and it is equivalent to the rule-3372206272 CA.

We can consider that this rule is an extension of the SlS model to high-speed motion since cars which stop at procedure (a) cannot move at procedure (b). Let us consider a relation between the SlS and EBCA1 models in detail. If we write each procedure of EBCA1 explicitly using an intermediate time step, we obtain

$$U_j^{t+1/2} = U_j^t + b_{j-1}^t - b_j^t \tag{6}$$

$$\begin{aligned} U_j^{t+1} = U_j^{t+1/2} &+ \min(U_{j-1}^{t+1/2} - \{U_{j-1}^t - b_{j-1}^t\}, L - U_j^{t+1/2}) \\ &- \min(U_j^{t+1/2} - \{U_j^t - b_j^t\},\ L - U_{j+1}^{t+1/2}). \end{aligned} \tag{7}$$

Equations (6) and (7) represent the above procedures (a) and (b), respectively, and $U_j^{t+1/2}$ denotes the car number at site j just after procedure (a). If we replace $t+1$ by $t+1/2$ in (1), we obtain (6). Moreover, if we replace t by $t+1/2$ and $t-1$ by t in (3), we obtain (6). Therefore we can consider EBCA1 to be a 'combination' of the BCA and SlS rules.

3. Fundamental diagrams and multiple states

We discuss fundamental diagrams of the new CA models described in the previous section. In the following, we will consider a periodic road or a circuit. All models in section 2 can be expressed in a conservation form such as

$$\Delta_t\, U_j^t + \Delta_j\, q_j^t = 0 \tag{8}$$

where Δ_t and Δ_j are the forward difference operator with respect to the indicated variable, and q_j^t represents a traffic flow. Average density ρ and average flow Q^t over all sites are defined by

$$\rho \equiv \frac{1}{KL}\sum_{j=1}^{K} U_j^t \qquad Q^t \equiv \frac{1}{KL}\sum_{j=1}^{K} q_j^t \tag{9}$$

where K is the number of sites in a period. Since all models are in a conservation form, average density does not depend on time and we can use ρ without a superscript t.

Figures 2(*a*)–(*d*) are density-flow diagrams of the previous models with $L = 2$ and $K = 30$. Figures 2(*a*)–(*d*) correspond to the multi-value QS, multi-value SlS, EBCA2 and EBCA1 models, respectively. If we set a number N of cars, we obtain a unique density $\rho = N/KL$ but there are many initial distributions of cars. Each initial distribution does not always reach a steady flow and often makes a periodic state. Q^t can also change periodically in time. Therefore, we plot every point in the figures by averaging Q^t from $t = 2$ to 4 K. (We use Q as an average flow.) Note that $t = 2$ K is long enough from the initial time to obtain a periodic state and 2 K time steps is much longer than its period.

Moreover, we obtain different values of Q from different initial distributions for a given ρ in figures 4(*b*)–(*d*). In [16], this type of state, giving different Q values for the same density, is called 'multiple state' [16]. We use this terminology for other models in this paper. Multiple states exist around a critical density. Especially for the multi-value SlS model (figure 2(*b*)) and the EBCA1 model (figure 2(*d*)), we see a thick branch other than lines forming an inverse λ shape and the distribution of data in the branch look random. This fact is interesting because evolution rules are completely deterministic. In both models, the congested states of high flow rate cover a two-dimensional region in the diagram, which is also seen in figure 1.

Next, we focus on the EBCA1 model and examine its properties in detail. Figure 3 is a skeleton diagram of the EBCA1 model. Branches in figure 3 are drawn as straight lines and we obtain them using specific initial conditions. For example, states '$\cdots$ 111020111020 $\cdots$', '$\cdots$ 121212 $\cdots$', '$\cdots$ 002002 $\cdots$' and '$\cdots$ 222222 $\cdots$' are all steady states in the model, and

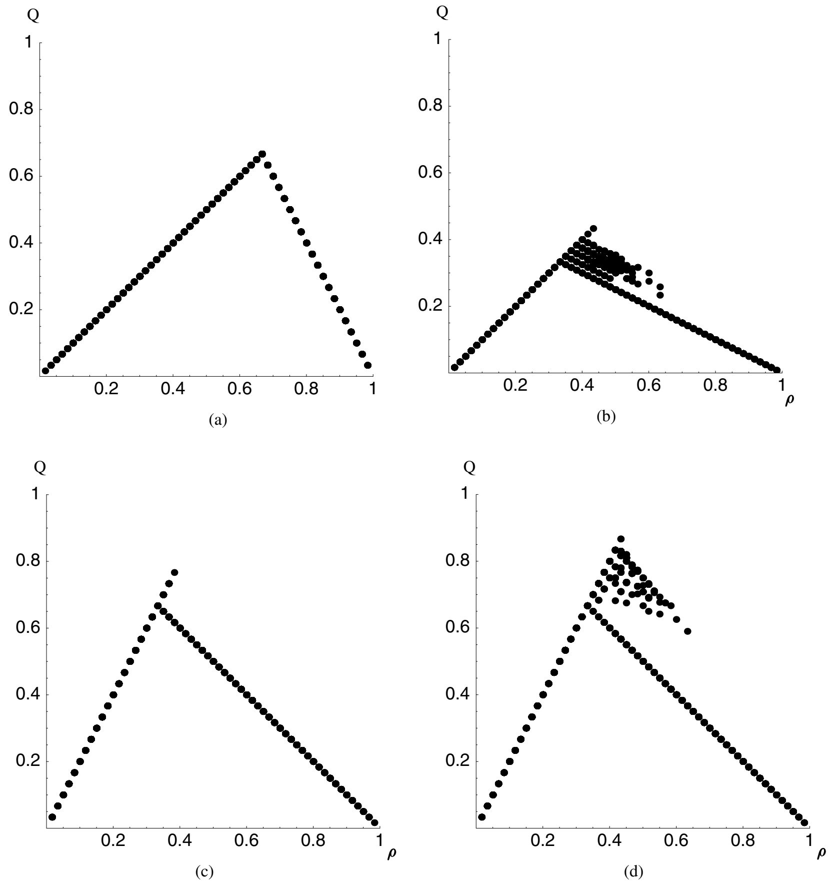

Figure 2. Fundamental diagrams of multi-value CA models with $L = 2$ and $K = 30$: (*a*) multi-value QS, (*b*) multi-value SlS, (*c*) EBCA2, (*d*) EBCA1 models.

flows of these states are plotted on B, C, D and E, respectively, in figure 3. There are two branches in the congested phase of higher density, that is, B–C and D–E. We call the branch D–E and B–C congested phase zero and phase one, respectively. It is noted that there are many states not on the skeleton in figure 3. For example, state $\cdots 211211 \cdots$ is a steady state of the model, and it has density $2/3$ and flow $1/2$.

Figure 4 shows a time evolution of flow Q^t in the EBCA1 model with a density of phase transition region. The number of total sites K is 60 and 240 in figures 4(*a*) and (*b*), respectively. In these figures, we can see periodic oscillation of flow appearing soon after an initial time. We observe that the maximum period of oscillation becomes longer as the system size becomes larger. Moreover, oscillation in a period is not simple, as shown in the figures. Figure 4(*c*) shows a power spectrum I versus frequency f defined by $I = |\sum_{t=0}^{T-1} Q^t \exp(2\pi \mathrm{i} f t/T)|/T$. We set $K = 240$ and $T = 1000$. This indicates that the irregular oscillation is similar to white noise.

Figure 5 is a fundamental diagram of EBCA1 with $L = 1$. It is interesting that there exist multiple states even in the case $L = 1$, which is equivalent to the rule-3372206272 CA. Among deterministic two-value CA models, only the SlS model is known to show metastable states so far. Thus, EBCA1 with $L = 1$ becomes the second example of such a kind of two-value CA.

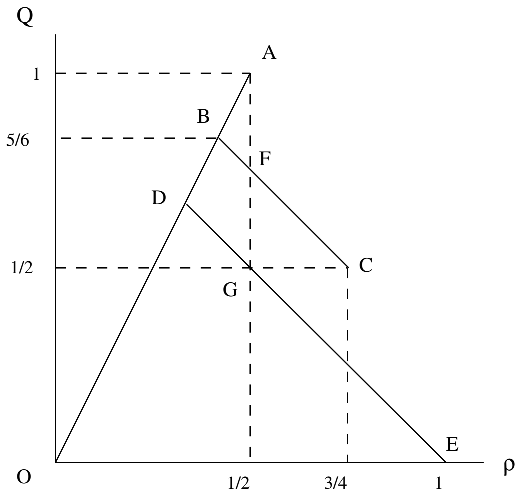

Figure 3. Schematic fundamental diagram of the EBCA1 model.

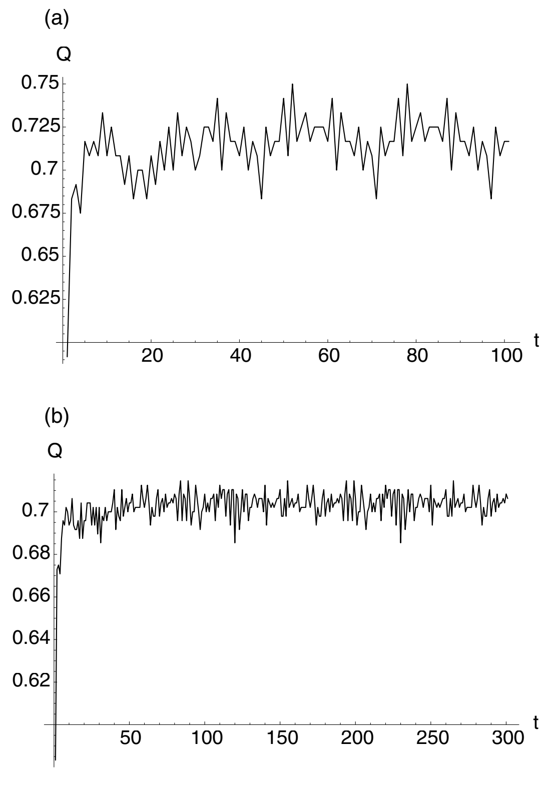

Figure 4. Time evolution of flow in the EBCA1 model. (*a*) $K = 60$, (*b*) 240, (*c*) power spectrum I versus frequency f of traffic flow for $K = 240$.

At the end of this section, we give a comment on the case of $L > 3$. The fundamental diagram of the multi-value QS model is the same, regardless of L. In other models, small

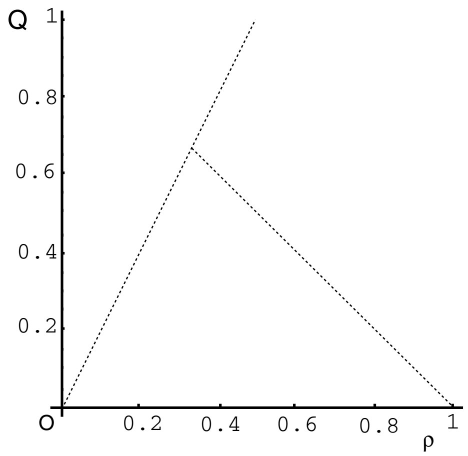

Figure 5. Fundamental diagram of the EBCA1 model with $L = 1$.

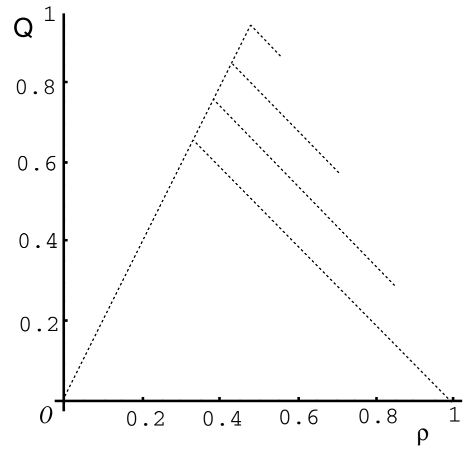

Figure 6. Fundamental diagram of the EBCA2 model with $L = 7$.

branches increase around the critical density as L becomes larger. Let us show this phenomenon using EBCA2 which has clear branches. The fundamental diagram of EBCA2 with $L = 7$ is given in figure 6. There are many branches around the critical density. We can give a partial explanation of these branches. Let us assume initial values on all sites are restricted to n or $L - n$, where $0 \leqslant n < L/2$. Then, if we use a transformation $V = (U - n)/(L - 2n)$ from U to V, $V = 0$ corresponds to $U = n$ and $V = 1$ to $U = L - n$. We can easily show that the evolution equation on V is obtained by replacing U by V and L by 1 in (4). Equation (4) with $L = 1$ is the FI model itself and it is a two-value CA. Therefore, if all initial values of U are restricted to n or $L - n$, then U at any site is always n or $L - n$ and the evolutionary state is the same as that of the FI model by replacing 0 by n and 1 by $L - n$.

The above fact implies a shape of the fundamental diagram. Consider an arbitrary state of the FI model, that is, (4) with $L = 1$. Let us assume the density is ρ and the average flow is Q for that state. We can obtain a corresponding state of (4) with $L > 0$ by replacing 0 by n and 1 by $L - n$. Then, using (4), we can easily show that the density of that state is $(1 - \frac{2n}{L})\rho + \frac{n}{L}$

and the flow is $(1 - \frac{2n}{L})Q + \frac{2n}{L}$. Moreover, we can also show that a fundamental diagram of the FI model is exactly given by two segments whose end points are $(0, 0)$, $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{1}{3}, \frac{2}{3})$, $(1, 0)$, respectively. Therefore, a diagram of EBCA2 with $L > 0$ has at least a segment with end points $(\frac{n}{L}, \frac{2n}{L})$ and $(\frac{1}{3} + \frac{n}{3L}, \frac{2}{3} + \frac{2n}{3L})$ and that with $(\frac{1}{3} + \frac{n}{3L}, \frac{2}{3} + \frac{2n}{3L})$ and $(1 - \frac{n}{L}, \frac{2n}{L})$ for any n $(0 \leqslant n < L/2)$. Figure 6 is a $L = 7$ case and the shape of the diagram is strictly a superposition of all these segments.

4. Stability of flow

In this section, we investigate the stability of the branches in the fundamental diagram obtained in the previous section. First, we study a uniform state $\cdots 11111 \cdots$, which gives the maximum flow in each model. Let us define a weak perturbation by a perturbation changing a local state 11 to 20, and a strong perturbation by a perturbation changing 1111 to 2200. Both perturbations clearly do not change the density. These perturbations mean that a car or two cars in the uniform flow suddenly decrease their speed and consequently fully occupied sites '2' appear.

Figures 7(*a*)–(*c*) show an instability of uniform flow by a weak perturbation in the multi-value SlS, EBCA2 and EBCA1 models, respectively. We can observe a stop-and-go wave propagating backward from the perturbed site in figure 7(*a*) (multi-value SlS) and (*c*) (EBCA1), while the wave does not appear in (*b*) (EBCA2). In figure 7(*c*), the flow Q decreases by the weak perturbation and transits from A to F in figure 3. Moreover, we see that the final steady state contains locally two states corresponding to B and C in figure 3. The flow of state B is $5/6$ and that of C is $1/2$, both states are in phase one, and they give the maximum and minimum flow in that phase. Numerical results from various initial states with a weak perturbation show that a state of higher density tends to give this type of extreme local state on a branch after some time steps. Moreover, if we give a strong perturbation of uniform state $\cdots 111 \cdots$ corresponding to A, it goes down directly to G$(1/2, 1/2)$ in phase zero. Figure 7(*d*) shows an instability due to a strong perturbation in the EBCA1 model. In this case, a stop-and-go wave does not occur. Moreover, we see that the final state consists of two local states $\cdots 200200 \cdots$ and $\cdots 222222 \cdots$, corresponding to D and E in figure 3, which also give the maximum and minimum flow in phase zero, respectively.

Since a perturbed uniform state becomes a congested one which consists of states giving the maximum and the minimum flow in a phase, we can predict a length of the final congested bunch. Assume α denotes a ratio of the length of the final congested bunch to the length of total sites. The density of B, F and C are $5/12$, $1/2$ and $3/4$, respectively. Thus we obtain

$$\tfrac{5}{12}(1 - \alpha) + \tfrac{3}{4}\alpha = \tfrac{1}{2}$$

and α becomes $1/4$, which coincides with that of figure 7(*c*). In the case of phase zero, we also obtain $\alpha = 1/4$ (figure 7(*d*)).

In contrast to the above facts, there also exist some states on branches stable against a weak perturbation. Let us consider states on branch A–D in figure 3. There exists a state $\cdots 011011011 \cdots$ on the branch and we obtain $\cdots 011020011 \cdots$ by a weak perturbation. Against the perturbation, the perturbed state is stable and it is still on the branch A–D. We call this type of state a metastable state. Moreover, let us consider a state $\cdots 111120111120 \cdots$ corresponding to F in figure 3. We can obtain $\cdots 201120111120 \cdots$ by a weak perturbation, and its flow does not change. Therefore, metastable states exist not only on branch A–D but also on the phase one branch. In these cases, the effect of the site '2' does not spread over the entire system during the time evolution. Thus some part of the multiple branches are metastable states, which can be observed in long time simulations starting from random initial conditions.

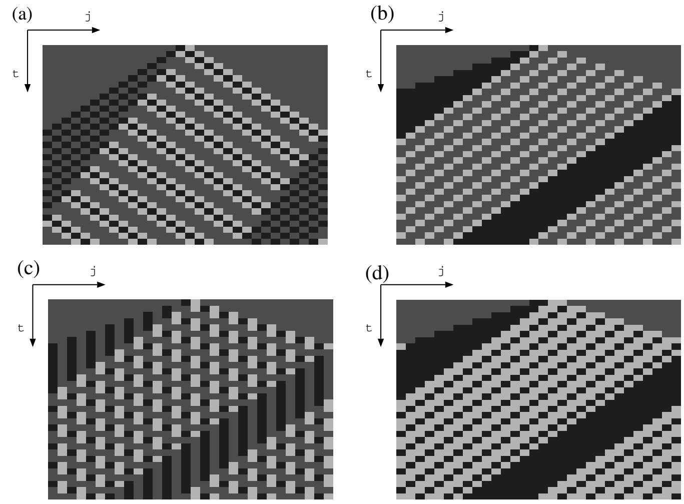

Figure 7. Instability of uniform flow due to a weak perturbation. (*a*) Multi-value SlS, (*b*) EBCA2, (*c*) EBCA1 models. (*d*) shows an instability due to a strong perturbation in the EBCA1 model. In all figures, black, dark gray and light gray squares denote a value 2, 1 and 0, respectively.

5. Concluding discussions

In this paper, some multi-value generalizations of the rule-184 family are studied by using the max-plus representation. We obtain evolution equations in a conserved form giving $(L+1)$-value CA. We have proved that the EBCA1 model is a generalization of the SlS model, and it gives multiple states and stop-and-go waves. We consider that a suitable traffic model gives those two phenomena at least. If we observe real traffic data like in figure 1, the existence of multiple states is clear. In real traffic, cars are considered to be always perturbed by some traffic effects. Therefore, we consider the existence of stable branches against a weak perturbation is also desirable for a model. Moreover, stop-and-go waves are often observed in real congested traffic. Considering these points, EBCA1 is the most suitable model among the models of this paper.

Next, we give a presumption on a fundamental diagram of real data. In figure 1, we see the multiple state clearly in an acceleration lane. In a driving lane, since the average speed of cars is slower than in the acceleration lane, flow tends to be stable and the branch is not clear. In the acceleration lane, drivers move faster than cars in the driving lane, then over-dense free flow will be likely to occur and we can see the multiple state around a critical density in the diagram. Moreover, from the stability analysis we find that there are metastable states even in the congested state in the EBCA1 model. Although we cannot clearly see the fact due to the fluctuation of data in figure 1, we expect that there exist both a metastable 'weak jam' (line B–C) and a 'strong jam' (line D–E) in real highway traffic. The weak jam seems to closely resemble the synchronized flow introduced by Kerner [17, 18]. He observed that there are three traffic phases: (1) free flow, (2) synchronized flow and (3) wide moving jams. The jams emerge in free flow due to a sequence of two phase transitions: 'free flow → synchronized flow → jams'. This is similar to the case of EBCA1, and all other CA models [5, 6, 9] cannot explain the existence of the congested state of high flow rate.

Finally, we point out future problems. We have investigated mainly models with $L = 2$. On multiple branches due to larger L, only EBCA2 gives clear branches and we can easily

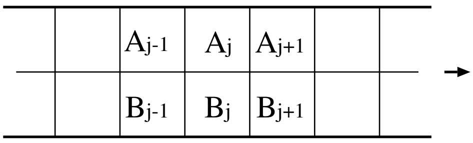

Figure 8. Interpretation of BCA with $L = 2$ as a model of coupled single lanes.

discuss their stability. EBCA1 will show two-dimensional complicated branches in general L, and the analysis on the phase transition becomes complicated. We should solve this difficulty, including how we choose the value L, because we consider the EBCA1 model is more suitable for a traffic model.

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Appendix

In this appendix, we show that the BCA model with $L = 2$ can be interpreted as a two-lane model of the rule-184 CA. Let us call one of the two lanes the A-lane and the other the B-lane. Both lanes are divided into discrete sites as shown in figure 8. Assume A_j^t and B_j^t denote the number of cars at site j and time t in the A-lane and B-lane, respectively. Since all sites can hold only one car at most, A_j^t and B_j^t are always either 0 or 1. A car at site j in the A-lane moves according to the following rules:

(a) If site $j + 1$ in the A-lane is empty, the car moves to that site.
(b) If site $j + 1$ in the A-lane is not empty and sites j and $j + 1$ in the B-lane are both empty, the car can move to site $j + 1$ in the B-lane.
(c) Otherwise, the car stays at site j in the A-lane.

As for a car in the B-lane, a symmetrical rule with respect to lane symbol is applied. The above rule can be interpreted as follows: every car moves in its own lane prior to the other lane (rule (a)). However, if a car cannot move in its own lane and only if there is no traffic in the other lane, it changes lane and moves forward in the other lane (rule (b)). Since cars move independently in each lane according to rule-184 if rule (b) is omitted, the above rule can be interpreted as a two-lane model based on rule-184 with lane-changing effect.

We can express the rule by a couple of evolution equations as follows:

$$\begin{aligned} A_j^{t+1} = A_j^t &+ \min(A_{j-1}^t, 1 - A_j^t) - \min(A_j^t, 1 - A_{j+1}^t) + \min(1 - A_{j-1}^t, 1 - A_j^t, B_{j-1}^t, B_j^t) \\ &- \min(A_j^t, A_{j+1}^t, 1 - B_j^t, 1 - B_{j+1}^t) \end{aligned} \tag{10}$$

$$\begin{aligned} B_j^{t+1} = B_j^t &+ \min(B_{j-1}^t, 1 - B_j^t) - \min(B_j^t, 1 - B_{j+1}^t) + \min(1 - B_{j-1}^t, 1 - B_j^t, A_{j-1}^t, A_j^t) \\ &- \min(B_j^t, B_{j+1}^t, 1 - A_j^t, 1 - A_{j+1}^t). \end{aligned} \tag{11}$$

The last two terms in both equations express the lane-changing effect. If those terms are omitted, both equations become independent of each other and are equivalent to the rule-184 CA (BCA model with $L = 1$). If U_j^t is defined by

$$U_j^t = A_j^t + B_j^t \tag{12}$$

it denotes a sum of cars on both lanes. Since A_j^t and B_j^t is always 0 or 1, we can derive an evolution equation on U_j^t from (10) and (11) as follows:

$$U_j^{t+1} = U_j^t + \min(U_{j-1}^t, 2 - U_j^t) - \min(U_j^t, 2 - U_{j+1}^t). \tag{13}$$

This equation is equivalent to BCA with $L = 2$.

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